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Fermionic and supersymmetric stochastic processes

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The fermionic Fock space is represented by the Wiener chaos. This identification allows one to define fermionic Brownian motion with a probability measure. In the underlying geometrical picture this Brownian motion evolves in the linear space of the generators of the Grassmann algebra which spans the Fock space. More general stochastic processes can be derived with the help of stochastic differential equations. The generalization to supersymmetric processes is based on the Wiener–Grassmann product of Le Jan, an algebraic structure which is adequate to investigate differential operators on Wiener spaces.

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1. Introduction

Fermionic and supersymmetric processes have been introduced as a useful analytic tool in elementary particle physics by Haba [1] and by Rogers [2,3]. In these approaches the usual stochastic calculus is transferred algebraically to a Grassmann algebra or to a superalgebra without reference to a probability measure. The aim of this lecture is to present a fermionic and to a lesser extent also a supersymmetric stochastic calculus which combines genuine probability with the non-commutative geometrical structure of a Grassmann algebra or a superalgebra. This approach is motivated by Euclidean quantum field theory on one side and by recent developments of a non-commutative stochastic calculus in the mathematics literature [4–7] on the other side. The supersymmetric version is closely related to the investigation of differential forms on Wiener spaces.

2. Multiple Wiener–Itô integrals and chaos expansion

Let $B_t(\omega)$, $t\in\mathbb{R}_+$, be the one-dimensional Brownian motion with continuous trajectories $\omega(t)$, $t\geqslant 0$, starting at the origin at time zero. The corresponding probability measure on $\Omega = C(\mathbb{R}_+)$ is the Wiener measure. The normalization is given by the expectation $EB_{t_1}B_{t_2} = \min(t_1, t_2)$ if $t_1, t_2 \geqslant 0$. Following the basic work of Wiener about homogeneous chaos [8] Itô has defined in ref. [9] multiple stochastic integrals $\int F(t_1, ..., t_n)\, \mathrm{d}B_{t_1} ... \mathrm{d}B_{t_n}$ for symmetric functions $F(t_1, ..., t_n) \in \mathscr{L}^2(\mathbb{R}^n_+)$. But the actual calculation of such integrals reduces to a calculation of an iterated integral on the open simplex $\Delta^n = \{(t_1, ..., t_n) \,|\, 0 < t_1 < \cdots < t_n\} \subset \mathbb{R}^n_+$, see, e.g., ref. [4]. Let $F(t_1, ..., t_n)$ be a numerical function on the open simplex Δ^n, square integrable with respect to the Lebesgue measure, i.e., $F \in \mathscr{L}^2(\Delta^n)$. Then the Itô prescription to evaluate

$$\phi_F(\omega) = \int_{\Delta^n} F(t_1, ..., t_n)\, \mathrm{d}B_{t_1} \cdots \mathrm{d}B_{t_n} \tag{2.1}$$

leads to a function $\phi_F(\omega)$ on the probability space Ω which is square integrable with respect to the Wiener measure μ, i.e. $\phi_F(\omega) \in \mathscr{L}^2_\mu(\Omega)$. Moreover the mapping $F \to \phi_F$ is an isometric injection $\mathscr{L}^2(\Delta^n) \to \mathscr{L}^2_\mu(\Omega)$ with

$$E|\phi_F|^2 \equiv \int |\phi_F(\omega)|^2\, \mathrm{d}\mu(\omega) = \int_{\Delta^n} |F(t_1, ..., t_n)|^2\, \mathrm{d}t_1 \cdots \mathrm{d}t_n\,.$$

The closed subspace of $\mathscr{L}^2_\mu(\Omega)$ which is the image (2.1) of $\mathscr{L}^2(\Delta^n)$ is called the nth Wiener chaos and is denoted by $\mathbf{H}_n$. A closer inspection shows $\mathbf{H}_m \perp \mathbf{H}_n$ if $m \neq n$ and the direct orthogonal sum $\mathbf{H} \equiv \bigoplus_{n=0}^{\infty} \mathbf{H}_n$ spans the whole of $\mathscr{L}^2_\mu(\Omega)$. Here $\mathbf{H}_0$ is the set of constant functions on Ω.

The restriction of the range of integration in (2.1) to the open simplex is the essential starting point for the construction of fermionic stochastic variables. Any function $F(t_1, ..., t_n) \in \mathscr{L}^2(\Delta^n)$ can be extended either to a totally symmetric function $F_\mathrm{s}(t_1, ..., t_n) \in \mathscr{L}^2(\mathbb{R}^n_+)$ or to a totally antisymmetric function $F_\mathrm{a}(t_1, ..., t_n) \in \mathscr{L}^2(\mathbb{R}^n_+)$. In the usual bosonic stochastic calculus only the identification with the symmetric function is considered. But it is as well possible to choose the identification with the antisymmetric function. That will lead to the fermionic calculus presented in the following section.

Let $\mathscr{X} = \mathscr{L}^2(\mathbb{R}_+)$ be the space of real functions on $\mathbb{R}_+$ square integrable with respect to the Lebesgue measure. The (anti)symmetric tensors of rank n of the space $\mathscr{X}$ correspond to (anti)symmetric functions $F(t_1, ..., t_n) \in \mathscr{L}^2(\mathbb{R}^n_+)$ which are uniquely determined by their values on the open simplex Δ^n. The Fock space $\mathscr{F}(\mathscr{X})$ of (anti)symmetric tensors of $\mathscr{X}$ is therefore the linear space of all sequences

$$F = \{F_0, F_1(t), F_2(t_1, t_2), ...\} \tag{2.2}$$

of functions $F_0 \in \mathbb{R}$ and $F_n(t_1, ..., t_n) \in \mathscr{L}^2(\Delta^n)$, $n = 1, 2, ...$, with the norm $\|F\|$ given by

$$\|F\|^2 = |F_0|^2 + \sum_{n=1}^{\infty} \int_{\Delta^n} |F_n(t_1, ..., t_n)|^2 \, dt_1 \cdots dt_n . \tag{2.3}$$

Following Guichardet [10] the Fock space vector F can be written as function $F(T)$ on the power set of $\mathbb{R}_+$, $T \in \mathscr{P}(\mathbb{R}_+) = \{\text{finite subsets of } \mathbb{R}_+\}$. Any subset $T \subset \mathbb{R}_+$ with $|T| = n$ elements, $T = \{t_1, ..., t_n\}$, corresponds uniquely to a point $(t_1, ..., t_n) \in \Delta^n$. With the identification $F(T) = F_n(t_1, ..., t_n)$ if $T = \{t_1, ..., t_n\}$ and $F(\emptyset) = F_0$, the whole sequence (2.2) is represented by $F(T)$, $T \in \mathscr{P}(\mathbb{R}_+)$. The equation (2.3) can be written in the short notation

$$\|F\|^2 = \int_{\mathscr{P}(\mathbb{R}_+)} |F(T)|^2 \, dT \tag{2.4}$$

and the Wiener isomorphism $F \in \mathscr{F}(\mathscr{X}) \to \phi_F \in \mathscr{L}^2_\mu(\Omega)$,

$$\phi_F(\omega) = F_0 + \int_{\mathbb{R}_+} F_1(t) \, dB_t + \int_{\Delta^2} F_2(t_1, t_2) \, dB_{t_1} \, dB_{t_2} + \cdots \tag{2.5}$$

takes the condensed form

$$\phi_F(\omega) = \int_{\mathscr{P}(\mathbb{R}_+)} F(T) \, dB_T . \tag{2.6}$$

Let $\mathscr{A}(\mathscr{X})$ be the linear span of all tensors of finite rank. Then the image (2.6) of $\mathscr{A}(\mathscr{X})$ is the algebraic sum $\oplus_n \mathbf{H}_n$. On $\mathscr{A}(\mathscr{X})$ we can define as usual the symmetric tensor product $F, G \in \mathscr{A}(\mathscr{X}) \to F \vee G \in \mathscr{A}(\mathscr{X})$ which is generated by

$$(f_1 \vee \cdots \vee f_n)(t_1, ..., t_n) := \sum_I f_{i_1}(t_1) \cdots f_{i_n}(t_n) \tag{2.7}$$

for arbitrary vectors $f_i \in \mathscr{X}$, $i = 1, 2, \ldots$. The summation extends over all permutations $I = (i_1, ..., i_n)$ of $(1, 2, ..., n)$. The basic product for a bosonic stochastic calculus is the slightly more complicated symmetric Wiener product, see, e.g., ref. [4]. The symmetric Wiener product $F, G \in \mathscr{A}(\mathscr{X}) \to F \nabla G \in \mathscr{A}(\mathscr{X})$ is generated by $f \nabla g = f \vee g + (f, g)$, where $(f, g) = \int_0^\infty f(t) g(t) \, dt$ is the inner product of $\mathscr{X}$. The Wiener isomorphism relates this product exactly to the numerical multiplication of functions in $\mathscr{L}^2_\mu(\Omega)$,

$$\phi_F(\omega) \cdot \phi_G(\omega) = \int (F \nabla G)(T) \, dB_T . \tag{2.8}$$

It is therefore the starting point of all calculations in the bosonic calculus. The essential problem of a fermionic calculus is now to define an antisymmetric counterpart of the symmetric Wiener product.

3. Bosonic and fermionic Brownian motion

To formulate a theory of a Markovian fermionic Brownian motion we have to extend the degrees of freedom. We start from a Brownian motion in a Euclidean space $\mathscr{E}=\mathbb{R}^{2N}$ of dimension $2N$, $N\in\mathbb{N}$. The corresponding probability space – the $2N$-fold Cartesian product of that of one-dimensional Brownian motion – is again denoted by Ω. The inner product of $\mathscr{E}$ is a positive bilinear symmetric form $f, g\in\mathscr{E}\to\langle f\,|\,g\rangle\in\mathbb{R}$. For the fermionic calculus we choose an orthogonal mapping $j:\mathscr{E}\to\mathscr{E}$ with $j^2=-\mathrm{id}$. Then

$$\gamma(f,g):=\langle f\,|\,jg\rangle \tag{3.1}$$

is a skew-symmetric bilinear form on $\mathscr{E}$. The basic Hilbert space of both the bosonic and the fermionic calculus is $\mathscr{X}=\mathscr{L}^2(\mathbb{R}_+,\mathscr{E})=\mathscr{L}^2(\mathbb{R}_+)\otimes\mathscr{E}$ with the inner product $(f\,|\,g)=\int\langle f(t)\,|\,g(t)\rangle\,\mathrm{d}t$. The space $\mathscr{F}_n(\mathscr{X})$ of (anti)symmetric tensors of rank n is given by square integrable functions $(t_1,\dots,t_n)\in\Delta^n\to F(t_1,\dots,t_n)\in\mathscr{E}^{\otimes n}$. The restriction $(t_1,\dots,t_n)\in\Delta^n$ accounts again for the possibility to extend $F(t_1,\dots,t_n)$ either to a symmetric or to an antisymmetric tensor in $\mathscr{X}^{\otimes n}$. The (anti)symmetrization does not determine the values on the diagonals $t_i=t_j$, $i\neq j$, but this subset of $\mathbb{R}^n_+$ is of Lebesgue measure zero. The inner product of the Fock space $\mathscr{F}(\mathscr{X})$ is

$$\begin{aligned}(F\,|\,G)&=\int\langle F(T)\,|\,G(T)\rangle\,\mathrm{d}T\\&=\sum_{n=0}^{\infty}\int_{\Delta^n}\langle F_n(t_1,\dots,t_n)\,|\,G_n(t_1,\dots,t_n)\rangle_n\,\mathrm{d}t_1\cdots\mathrm{d}t_n\,,\end{aligned}$$

where $\langle\cdot\,|\,\cdot\rangle_n$ is the usual inner product of the tensor space $\mathscr{E}^{\otimes n}$. The Wiener isomorphism $F\in\mathscr{F}(\mathscr{X})\to\phi_F(\omega)\in\mathscr{L}^2_\mu(\Omega)$ is given by

$$\begin{aligned}\phi_F(\omega)&=\int\langle F(T)\,|\,\mathrm{d}B_T\rangle\\&=F_0+\int_0^\infty\langle F_1(t)\,|\,\mathrm{d}B_t\rangle+\int_{\Delta^2}\langle F_2(t_1,t_2)\,|\,\mathrm{d}B_{t_1}\otimes\mathrm{d}B_{t_2}\rangle_2+\cdots\,.\end{aligned} \tag{3.2}$$

The symmetric tensor product (2.7) is easily generalized to

$$(f_1\vee\cdots\vee f_n)(t_1,\dots,t_n):=\sum_I f_{i_1}(t_1)\otimes\cdots\otimes f_{i_n}(t_n)\,, \tag{3.3}$$

and an appropriate choice of the antisymmetric tensor product is

$$(f_1\wedge\cdots\wedge f_n)(t_1,\dots,t_n):=\sum_I(\operatorname{sgn}I)f_{i_1}(t_1)\otimes\cdots\otimes f_{i_n}(t_n) \tag{3.4}$$

for arbitrary vectors $f_i\in\mathscr{X}$, $i=1,\dots,n$.

The numerical multiplication corresponds as in (2.8) to the symmetric Wiener product $\phi_F(\omega)\cdot\phi_G(\omega)=\int\langle (F\nabla G)(T)\,|\,\mathrm{d}B_T\rangle$ and the symmetric Wiener product $F\nabla G$ is again generated by

$$f\nabla g := f\vee g + (f,g)\,, \tag{3.5}$$

with the symmetric inner product (f,g) of $\mathscr{X}$. The product $F\nabla G$ is a bilinear symmetric and associative product on $\mathscr{A}(\mathscr{X})$, the linear span of all tensors of finite rank. This product is algebraically isomorphic to the symmetric tensor product.

The fermionic counterpart is the antisymmetric Wiener product $F\triangle G$, which is generated as associative product by

$$f\triangle g := f\wedge g + \omega(f,g) \tag{3.6}$$

with the antisymmetric tensor product (3.4) and the skew-symmetric form

$$\omega(f,g) := \int_0^\infty \gamma(f(t),g(t))\,\mathrm{d}t \tag{3.7}$$

on $\mathscr{X}$, which is derived from the skew-symmetric form γ of $\mathscr{E}$ in the same way as the inner product of $\mathscr{X}$ is derived from the inner product of $\mathscr{E}$. The antisymmetric Wiener product is defined on $\mathscr{A}(\mathscr{X})$ and it is algebraically isomorphic to the antisymmetric tensor product. With the Wiener isomorphism the antisymmetric Wiener product can be transferred to a new product on $\mathscr{L}^2_\mu(\Omega)$,

$$(\phi_F\times\phi_G)(\omega) := \int \langle (F\triangle G)(T)\,|\,\mathrm{d}B_T\rangle\,. \tag{3.8}$$

This new product will be called Grassmann product. For constant functions (vacuum sector) this product is defined as $1\times\phi_F=\phi_F\times 1=\phi_F$ if $F\in\mathscr{F}(\mathscr{X})$.

The antisymmetric Wiener product has been introduced by Le Jan [5] in a supersymmetric extension under the name Wiener–Grassmann product. An elucidating investigation of this product has been given by Meyer [4]. Its relation to Krée's theory of fermionic integration [11] has been studied in ref. [12], where the name Grassmann product has been used for $F\triangle G$.

By definition of the Wiener chaos we know $E\phi_F=(1\,|\,F)$ for any $F\in\mathscr{F}(\mathscr{X})$. Therefore the two following identities hold for $F, G\in\mathscr{A}(\mathscr{X})$:

$$E\phi_F\cdot\phi_G=(1\,|\,F\nabla G)\,,\qquad E\phi_F\times\phi_G=(1\,|\,F\triangle G)\,. \tag{3.9}$$

These identities allow one to relate algebraic bosonic or fermionic expressions to expectation values with the Wiener measure.

The definition of the antisymmetric Wiener product and consequently of the Grassmann product (3.8) is not unique, since the antisymmetric tensor product (3.4) and the skew-symmetric form (3.7) depend on a partial ordering in the space $\mathscr{X}$, which can be chosen in different ways, see refs. [4,12]. The choice (3.6) together with (3.4) and (3.7) has the advantage that it is compatible with the causal (Markovian) structure of Brownian motion. (In refs. [4,12] another convention has been used. Then an additional unitary transformation is needed for a causal white noise calculus, see eq. (4.27) of ref. [12].)

So far two algebraic structures have been defined on $\mathscr{L}^2_\mu(\Omega)$, which correspond either to a bosonic or to a fermionic calculus. But we have not yet introduced fermionic Brownian motion. To that end eqs. (3.9) are specialized to vectors $f, g \in \mathscr{X}$,

$$E\phi_f \cdot \phi_g = (f|g) , \qquad E\phi_f \times \phi_g = \omega(f, g) . \tag{3.10}$$

We choose an orthonormal basis of $\mathscr{E}$, $\{e_\mu, \mu = 1, ..., 2N\}$, $\langle e_\mu, e_\nu \rangle = \delta^{\mu\nu}$. The skew-symmetric form (3.1) is then determined by the matrix $\gamma(e_\mu, e_\nu) = C^{\mu\nu}$, which satisfies $C^{\mu\nu} = -C^{\nu\mu}$ and $C^2 = -I$. With the function $F_1(t) = f(t) = e_\mu \Theta(s-t) \in \mathscr{X}$, where $s \in \mathbb{R}_+$ is a time parameter, the chaos expansion (3.2) yields the μ-component of Brownian motion B^μ_s. The respective expectations (3.10) are

$$EB^\mu_{s_1} \cdot B^\nu_{s_2} = \delta^{\mu\nu} \min(s_1, s_2) , \qquad EB^\mu_{s_1} \times B^\nu_{s_2} = C^{\mu\nu} \min(s_1, s_2) , \tag{3.11}$$

for all $\mu, \nu = 1, ..., 2N$ and $s_1, s_2 \geqslant 0$. Fermionic Brownian motion is therefore the usual stochastic process which takes place in the first Wiener chaos, but it is considered as part of the infinite-dimensional Grassmann algebra based on the Grassmann product (3.8).

Remarks

1. In this lecture only stochastic processes with a finite number of components are considered. It is possible to extend the results presented here to infinite-dimensional processes or to random fields. For that purpose multiple stochastic integrals should be derived from a random orthogonal measure – in our case a white noise measure – as done, e.g., in ref. [13]. One may also start from the usual theory of Gaussian integration. But then bosonic and fermionic normal ordering prescriptions have to be discussed in some detail to relate the tensor products with the Wiener products [12].

2. Any element of the first chaos – including Brownian motion – is a multiplication operator in the respective bosonic or fermionic function algebra. In the Fock space picture the multiplication is given by the symmetric or the antisymmetric Wiener product. Their definitions yield the decompositions $f \nabla H = f \vee H + f \neg H$ or $f \triangle H = f \wedge H - (jf) \ulcorner H$ for any $f \in \mathscr{X}$ and $H \in \mathscr{A}(\mathscr{X})$. Here $f \neg H$ and $f \ulcorner H$ are the contractions or interior products with f in the respective tensor algebra (equivalent to the annihilation operators) and j is the skew-symmetric operator used to define (3.1). One can therefore decompose bosonic and fermionic Brownian motion into the creation and annihilation processes of Applebaum, Hudson and Parthasarathy [6,7]. On the other hand the fermionic creation and

the annihilation processes can be added (without the operator j) to the Clifford process of Barnett, Streater and Wilde [14]. An introduction to all these processes has been given in ref. [15].

4. Stochastic differential equations

4.1. LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

In the case of linear stochastic differential equations of the type $dZ_t = a(t)Z_t\,dt + b(t)\,dB_t$, where $a(t)$ and $b(t)$ are numerical coefficient functions, the process Z_t is a linear transform of Brownian motion. Under usual initial conditions Z_t is a Gaussian process in the first chaos and the bosonic or fermionic character shows up only in the calculation of products. The expectation values of products follow in agreement with (3.9) and they lead exactly to the combinatorics of n-point Schwinger functions of free bosonic or fermionic quantum fields. But in the Euclidean quantum field theory of fermions one is confronted with an additional difficulty: the two-point function should be the Green's function of a linear elliptic differential operator. As a consequence one has to use generalized stochastic processes with a strange involution. These problems already show up for one-dimensional complex processes related to the differential operators $D_\pm = m \pm d/dt$ with parameter $m \geqslant 0$. Let $D_+^{-1}(t_1, t_2)$ be the (retarded) Green's function of D_+. Then we are asking for a pair of (generalized) stationary Gaussian processes Z_t and Z_t^* which satisfy

$$EZ_{t_1} \times Z_{t_2} = EZ_{t_1}^* \times Z_{t_2}^* = 0\,, \qquad EZ_{t_1} \times Z_{t_2}^* = D_+^{-1}(t_1, t_2)\,. \tag{4.1}$$

The processes Z_t and Z_t^* are related by an involution, which is more complicated than complex conjugation. Since $D^{-1}(t, t)$ is not well defined at least one of the processes Z_t or Z_t^* has to be a generalized process. It should be emphasized that this singular behaviour is not a consequence of the fermionic character of the processes but of the singular correlation function. The two-dimensional Grassmann Brownian bridge of Rogers [2] corresponds to such a pair of complex processes (with parameter $m=0$ and antisymmetric Green's function). A possible solution of (4.1) with Markovian (generalized) processes is $Z_t = (D_+^{-1}\zeta)(t)$, the complex Ornstein–Uhlenbeck process, and $Z_t^* = \overline{\zeta(t)}$. Here $\zeta(t) = (d/dt)B_t$ is the complex white noise. But the spectral representation of stationary processes allows other, non-Markovian, solutions $Z_t = (D_+^{-1/2}\zeta)(t)$ and $Z_t^* = (D_-^{-1/2}\bar{\zeta})(t)$, which are more closely related to the random fields used for Euclidean Dirac spinors [16,17].

4.2. NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

All stochastic differential equations which satisfy the usual conditions for a strong solution allow an iteration with a chaos expansion. Therefore not only the solution but also the whole iteration series has a bosonic as well as a fermionic interpretation. Nice examples are given by Hida [18, sect. 4.6(iv)], who uses multiple Wiener integrals of the type (2.1) but without mentioning the fermionic interpretation.

There is a special class of stochastic differential equations which use the fermionic calculus explicitly and which correspond closely to the fermionic stochastic differential equations of Haba [1]. We start from a probability space of an N-dimensional complex Brownian motion B_t. Then any Grassmann polynomial

$$Z_t = \sum_{\mu,\nu} c_{\mu\nu}(t)\bar{B}_t^{\mu_1}\times\cdots\times\bar{B}_t^{\mu_m}\times B_t^{\nu_1}\times\cdots\times B_t^{\nu_n}\,,\quad m+n\leqslant N\,,$$

$$\mu=(\mu_1,\dots,\mu_m)\in\mathbb{N}^m\,,\quad \mu_1<\mu_2<\cdots<\mu_m\,,$$

$$\nu=(\nu_1,\dots,\nu_n)\in\mathbb{N}^n\,,\quad \nu_1<\nu_2<\cdots<\nu_n\,,$$

with smooth numerical functions $c_{\mu\nu}(t)$ is a process which is adapted to the filtration of B_t. Moreover, the differential of Z_t can be calculated with an Itô formula in which the partial derivatives are the antiderivations of the Grassmann algebra generated by the Grassmann product. Such a calculus has been proposed on an algebraic basis by Haba [1]. Consequently stochastic differential equations in which all terms are built up from Grassmann products, essentially agree with those presented in ref. [1]. But now the probabilistic interpretation is explicitly given. The connection with the calculus of Rogers is not so obvious.

5. Supersymmetric stochastic processes

There is a rather obvious generalization to supersymmetric processes. The space $\mathscr{E}$ of section 3 has to be extended to the sum of a bosonic and a fermionic space $\mathscr{E}=\mathscr{E}_B\oplus\mathscr{E}_F$ such that $\mathscr{X}=\mathscr{L}^2(\mathbb{R}_+,\mathscr{E}_B\oplus\mathscr{E}_F)=\mathscr{X}_B\oplus\mathscr{X}_F$ decomposes into $\mathscr{X}_B=\mathscr{L}^2(\mathbb{R}_+,\mathscr{E}_B)$ and $\mathscr{X}_F=\mathscr{L}^2(\mathbb{R}_+,\mathscr{E}_F)$. The Fock space is now $\mathscr{F}(\mathscr{X})=\mathscr{F}^+(\mathscr{X}_B)\otimes\mathscr{F}^-(\mathscr{X}_F)$. The $\pm$ signs indicate that the algebraic products on $\mathscr{F}^+(\mathscr{X}_B)$ are the symmetric tensor product and the symmetric Wiener product, whereas $\mathscr{F}^-(\mathscr{X}_F)$ is equipped with the antisymmetric products. Let $\mathscr{F}_p^-(\mathscr{X}_F)$ be the closed subspace of $\mathscr{F}^-(\mathscr{X}_F)$ spanned by all tensors of rank $p\in\{0,1,2,\dots\}$; then $\mathscr{F}(\mathscr{X})=\bigoplus_{p=0}^{\infty}\mathscr{F}_p(\mathscr{X})$ with $\mathscr{F}_p(\mathscr{X})=\mathscr{F}^+(\mathscr{X}_B)\otimes\mathscr{F}_p^-(\mathscr{X}_F)$ is a $\mathbb{Z}_2$ graded vector space. The parity of an element $F\in\mathscr{F}_p(\mathscr{X})$ is defined as $[F]=(1-(-1)^p)/2$. The basic tensor product of $\mathscr{F}(\mathscr{X})$ is the $\mathbb{Z}_2$ graded combination of (3.3) and (3.4),

$$(f_1\circ\cdots\circ f_n)(t_1,\dots,t_n)=\sum_I \sigma_I(f_1,\dots,f_n)f_{i_1}(t_1)\otimes\cdots\otimes f_{i_n}(t_n)\,. \tag{5.1}$$

The parity factor is

$$\sigma_I(f_1, ..., f_n) = (-1)^{N_I}, \quad N_I = \#\{(i_\mu, i_\nu) \mid \mu < \nu, i_\mu > i_\nu, [f_{i_\mu}] = [f_{i_\nu}] = 1\}.$$

The corresponding Wiener product, which is the Wiener–Grassmann product of Le Jan [5], is generated by $fg = f \circ g + \beta(f, g)$, where $\beta(f, g)$ is the supersymmetric bilinear form on $\mathscr{X}$ [19] derived from the symmetric inner product on $\mathscr{X}_B$ and the skew-symmetric form (3.7) on $\mathscr{X}_F$. Supersymmetric Brownian motion can be obtained from this product following the constructions of section 3. It is the tensor product of a bosonic Brownian motion and a fermionic Brownian motion. Infinitesimal supersymmetry transformations are antiderivations or superderivations [19] of this Wiener–Grassmann algebra. For the Brownian motion obtained from the product (5.1) these supersymmetry transformations do not include time derivatives. If we demand that the square of the supersymmetry operator is the generator of the process – as done in refs. [1] and [3] – then we need more complicated constructions where the fermionic part is of the type discussed at the end of section 4.1.

A final remark about the relation to geometry should be added. As already indicated in ref. [5], a $\mathbb{Z}_2$ graded tensor product of the type (5.1) is the adequate algebraic structure to investigate forms on infinite-dimensional linear spaces. The de Rham complex of differential operators presented by Arai in his lecture during this school, see also ref. [20], can be formulated in this language. The differential operators d_A and d_A^* of ref. [20] are in fact antiderivations of a Wiener–Grassmann algebra.

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